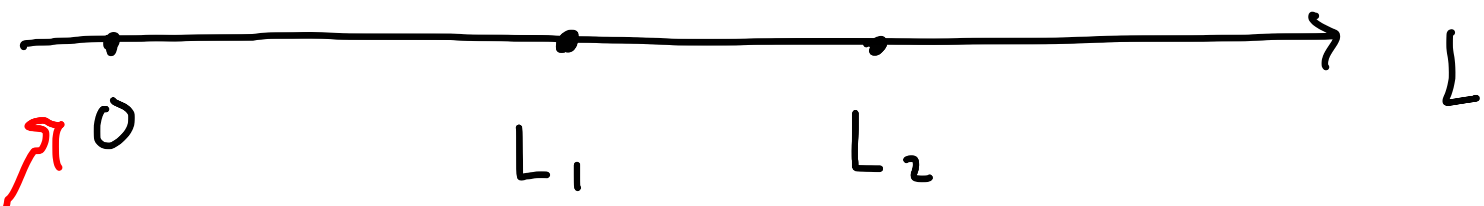
§6. Deformation Quantization and Algebraic Index

Last time :

Heat Kernel regularization



UV divergence = "counter-term"

$I[L_1]$ $\quad$ $I[L_2]$

At each $L > 0$, we have well-defined effective BV operator Δ_L

$\Rightarrow \quad (\mathcal{O}(\mathcal{E}), Q, \Delta_L)$ Effective DGBV

$I[L]$ solving the QME

- Δ_L = contraction w/. the smooth kernel K_L representing $e^{-L[Q,Q^+]}$

Let $\mathbb{H} = \{\varphi \in \mathcal{E} \mid [Q, Q^+]\varphi = 0\}$

$$= \{\varphi \in \mathcal{E} \mid Q\varphi = Q^+\varphi = 0\}$$

$$\simeq H^\bullet(\mathcal{E}, Q)$$

$\mathbb{H}$: Harmonics, also called "zero modes" which is finite dim'l.

∞-dim (-1)-symplectic $\xRightarrow{L\to\infty}$ finite dim (-1)-symplectic

$(\mathcal{E}, Q, \omega)$ $\qquad$ $(\mathbb{H}, \omega_H = \omega|_{\mathbb{H}})$

The BV operator Δ_H associated to ω_H^{-1} is

$$\Delta_H = \Delta_\infty$$

$$\Rightarrow \quad 0 \longrightarrow L=\infty$$

geometric data $\rightsquigarrow$ $I[\infty]$ solves QME for $(\mathcal{O}(\mathbb{H}), \Delta_H)$

Next Goal: Use the method we have discussed so far to do geometry & topology

We will explain two main examples

① 1d example: Topological Quantum mechanics and algebraic index

Ref:

- [Grady-Li-L]: Batalin-Vilkovisky quantization and algebraic index AIM (2017)
- [Gui-L-Xu]: Geometry of Localized effective theories, Exact semi-classical Approximation and the algebraic index CMP (2021)

② 2d example: Chiral CFT and Chiral index

Ref:

- [L]: Vertex algebras and quantum master equation JDG (2022)
- [Gui-L]: Elliptic trace map on chiral algebras arXiv: 2112.14572

· Deformation Quantization

Def'n. A Poisson manifold is a pair (X, P) where X is a smooth manifold, and $P \in \Gamma(X, \Lambda^2 TX)$ satisfying $\{P, P\}_{SN} = 0$.

Schouten-Nijenhuis bracket

P is called the Poisson tensor / bi-vector.

In local coordinates, we can write

$$P = \sum_{i,j} P^{ij}(x)\, \partial_i \wedge \partial_j$$

It defines a Poisson bracket $\{-,-\}_P$ on $C^\infty(X)$

$$\{f, g\}_P := \sum_{i,j} P^{ij}\, \partial_i f\, \partial_j g \quad \forall f, g \in C^\infty(X)$$

$\{P, P\}_{SN} = 0 \implies \{-,-\}_P$ satisfies Jacobi-identity

$\implies (C^\infty(X), \{-,-\}_P)$ Poisson algebra

Basic example: Let (X, ω) be a symplectic manifold.

$\omega = \frac{1}{2}\sum_{i,j} \omega_{ij}\, dx^i \wedge dx^j$ symplectic 2-form. Let

$$P = \omega^{-1} = \frac{1}{2}\sum_{i,j} \omega^{ij}\, \partial_i \wedge \partial_j \quad \text{where } (\omega^{ij}) = \text{inverse of } (\omega_{ij})$$

then $d\omega = 0 \iff \{P, P\}_{SN} = 0$

$\Longrightarrow$ (X, ω^{-1}) is a Poisson manifold

Deformation Quantization:

$$\begin{tikzcd} \text{Poisson} \arrow[r, bend left, "\text{quantization}"] & \text{Associative} \arrow[l, bend left, "\hbar \to 0"] \end{tikzcd}$$

[BFFLS]: Bayen, Flato, Fronsdal,
Lichnerowicz, Sternheimer

f $\qquad\qquad\qquad$ $\hat{f}$ = "operator"

Def'n: A Star-product on a Poisson manifold (X, P) is a $\mathbb{R}[[\hbar]]$-bilinear map

$$C^\infty(X)[[\hbar]] \times C^\infty(X)[[\hbar]] \longmapsto C^\infty(X)[[\hbar]]$$

$$f \times g \longmapsto f * g = \sum_{k\geq 0} \hbar^k C_k(f, g)$$

such that

① $*$ is associative: $(f*g)*h = f*(g*h)$

② $f*g = fg + O(\hbar) \quad \forall f, g \in C^\infty(X)$

③ $\frac{1}{2}(f*g - g*f) = \hbar\{f, g\} + O(\hbar^2), \ \forall f, g \in C^\infty(X)$

④ $C_k: C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ bidiff. operators

Then $(C^\infty(X)[[\hbar]], *)$ is called a deformation quantization of (X, P)

Existence of deformation quantization is highly nontrivial

- Symplectic case: Dewilde-Lecomte, Fedosov
- Poisson case: Kontsevich gives the complete solution for general Poisson manifold.

Example: [Moyal-Weyl Product] Let $X =$ w/.

symplectic form

$$\omega = \frac{1}{2}\sum_{i,j} \omega_{ij}\, dx^i \wedge dx^j \quad \text{w/. } \omega_{ij} = \text{Constant}$$

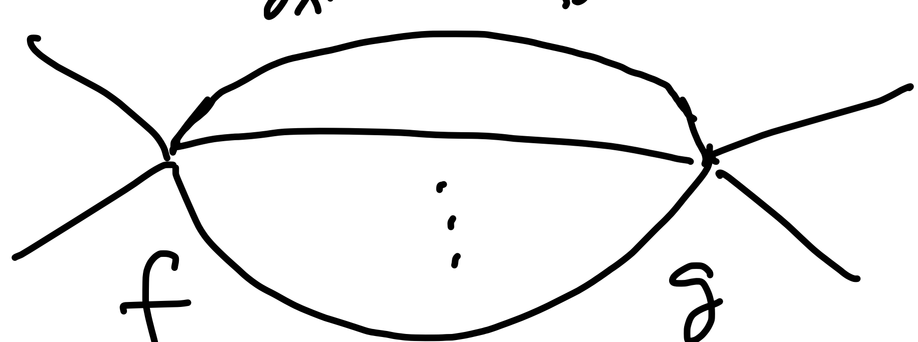
The Poisson tensor

$$P = \frac{1}{2}\sum_{i,j} \omega^{ij}\, \partial_i \wedge \partial_j$$

Given $f(x), g(x) \in C^\infty(\mathbb{R}^{2n})$, define the Moyal-Weyl product $*$ by

$$(f * g)(x) = \exp\left(\frac{\hbar}{2}\sum \omega^{ij} \frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)\Bigg|_{y=z=x} f(y)\, g(z)$$

$$\overleftarrow{\frac{\partial}{\partial x^i}}\, \omega^{ij}\, \overrightarrow{\frac{\partial}{\partial x^j}}$$



Then $*$ defines a deformation quantization.

RK. If $\omega^{ij} \neq$ const, then the above formula doesn't work.

For convenience, we can describe a formal version:

Let (V, ω) linear symplectic. $V \cong \mathbb{R}^{2n}$.

$$\omega : \wedge^2 V \mapsto \mathbb{R} \quad \text{symplectic pairing}$$

Write $\hat{\mathcal{O}}(V) := \widehat{\mathrm{Sym}}(V^\vee) = \prod_{k \geq 0} \mathrm{Sym}^k (V^\vee)$

$V^\vee = \mathrm{Hom}(V, \mathbb{R})$ linear dual.

Then the Moyal-Weyl product defines an associative algebra

$$(\hat{\mathcal{O}}(V)[[\hbar]], *) \quad \text{(formal) Weyl algebra.}$$

On the other hand, let

$$\hat{\Omega}_V^{-\bullet} := \hat{\mathcal{O}}(V) \otimes \wedge^{-\bullet}(V^\vee)$$

Here $\hat{\Omega}_V^{-p} := \hat{\mathcal{O}}(V) \otimes \wedge^p (V^\vee)$

(formal) p-forms sit in degree $-p$.

Let $d_V : \hat{\Omega}_V^{-p} \longmapsto \hat{\Omega}_V^{-(p+1)}$ be de Rham differential

Let $\Pi = \omega^{-1} \in \Lambda^2 V$ be the Poisson tensor

$\iota_\Pi : \hat{\Omega}_V^{-\bullet} \longmapsto \hat{\Omega}_V^{-(\bullet-2)}$ contraction w/. Π.

Let $\Delta = \mathcal{L}_\Pi = [d_V, \iota_\Pi] : \hat{\Omega}_V^{-\bullet} \longmapsto \hat{\Omega}_V^{-(\bullet-1)}$

the Lie derivative

$\Rightarrow (\hat{\Omega}_V^{-\bullet}, \Delta)$ defines a BV algebra

Geometrically, this leads to Koszul-Brylinski Complex/Homology

Physically, this is the effective geometry on "zero modes" of topological quantum mechanics as we will see.

· Fedosov quantization

We will focus on symplectic manifolds now.

Fedosov: a simple and geometric construction of deformation quantization on symplectic cases.

(X, ω) symplectic manifold

<u>Def'n</u>: We define the **Weyl bundle**

$$\mathcal{W}(X) := \prod_{k \geq 0} \mathrm{Sym}^k (T^*X)[[\hbar]]$$

So at each pt $p \in X$, its fiber is

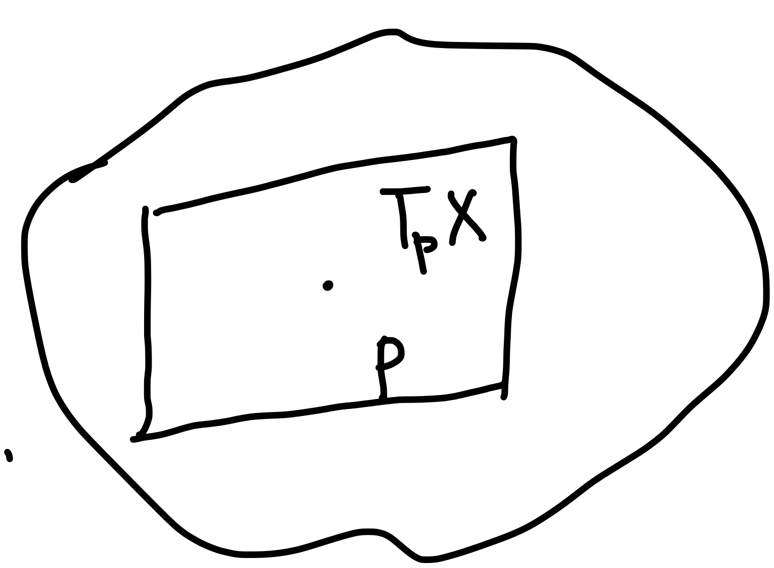
$$\mathcal{W}(X)\big|_p = \hat{O}(T_pX)[[\hbar]]$$

A local section of $\mathcal{W}(X)$ is

$$\sigma(x, y) = \sum_{k, \ell \geq 0} \hbar^k a_{k, i_1 \cdots i_\ell}(x) y^{i_1} \cdots y^{i_\ell}$$

where x: base coord. y: fiber coord.

$a_{k, i_1 \cdots i_\ell}(x)$ smooth functions



Since $(T_pX, \omega|_{T_pX})$ is linear symplectic, we have a fiberwise Moyal-Weyl product, still denoted by $*$

$\Rightarrow$ $(\mathcal{W}(X), *)$ ∞-dim bundle of algebras.

Let ∇ be a connection on TX which is torsion-free and compatible w/. ω $(\nabla\omega = 0)$. Such connection is called a symplectic connection. (exist and not unique)

∇ induces a connection on all tensors. In particular, It defines a connection on $\mathcal{W}(X)$, still denoted by ∇.

Its curvature is

$$\nabla^2\sigma = \frac{1}{\hbar}[R_\nabla, \sigma]_* \quad \forall\, \sigma \in \Gamma(X, \mathcal{W}(X))$$

where $R_\nabla = \frac{1}{4} R_{ijkl}\, y^i y^j dx^k \wedge dx^l \in \Omega^2(X, \mathcal{W}(X))$

$$R_{ijkl} = \omega_{im} R^m_{jkl}$$

Fedosov: Given a sequence $\{\omega_k\}_{k\geq 1}$ of closed 2-forms on X, there exists a unique (up to gauge) connection on $\mathcal{W}(X)$ of the form $\nabla + \frac{1}{\hbar}[\gamma, -]_*$ satisfying some initial condition and the equation

$$\nabla\gamma + \frac{1}{2\hbar}[\gamma,\gamma]_* + R_\nabla = \omega_\hbar$$

where $\omega_\hbar = -\omega + \sum_{k\geq 1} \hbar^k \omega_k$. (Fedosov eqn)

Let $D = \nabla + \frac{1}{\hbar}[\gamma, -]_*$. Then Fedosov Connection implies

$$D^2 = \frac{1}{\hbar}[\omega_\hbar, -]_* = 0.$$

central term

So we obtain a flat connection D on $\mathcal{W}(X)$

Let
$$W_D(X) := \{\sigma \in \Gamma(X, W(X)) \mid D\sigma = 0\}$$

be the space of flat sections. Then

$(W_D(X), *)$ is an associative algebra

Let $\sigma : W_D(X) \longmapsto C^\infty(X)[[\hbar]]$

by sending $y \longmapsto 0$ (symbol map)

Then σ is an isomorphism, and

$$f * g \longmapsto \sigma(\sigma^{-1}(f) * \sigma^{-1}(g))$$

defines a deformation quantization.

$\omega_\hbar$ is the corresponding characteristic class (moduli)

· Algebraic Index Theorem

Given a deformation quantization $(C^\infty(X)[[\hbar]], *)$

on a symplectic manifold w/. characteristic class $\omega_\hbar$.

there exists a unique trace map

$$\mathrm{Tr} : C^\infty(X)[[\hbar]] \longmapsto \mathbb{R}((\hbar))$$

Satisfying a normalization condition and the trace property

$$\mathrm{Tr}(f*g) = \mathrm{Tr}(g*f)$$

Then $\mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar} \hat{A}(X)$

This is the simplest version of algebraic index theorem formulated by Fedosov and Nest-Tsygan as the algebraic analogue of Atiyah-Singer Index Theorem.

We can similarly construct "deformation quantization" for $C^\infty(X, \mathrm{End}(E))[[\hbar]]$ and construct the trace map,

then $\mathrm{Tr}(1) = \int_X e^{\omega_\hbar/\hbar} \mathrm{ch}(E) \hat{A}(X)$

· Relation w/. QFT

In SUSY QFT, "localization" often appears

$$\int_{\mathcal{E}} e^{iS/\hbar} = \int_{\mathcal{M}} (-)$$

where $\mathcal{M} \subset \mathcal{E}$ is a finite dim'l space describing some interesting moduli space.

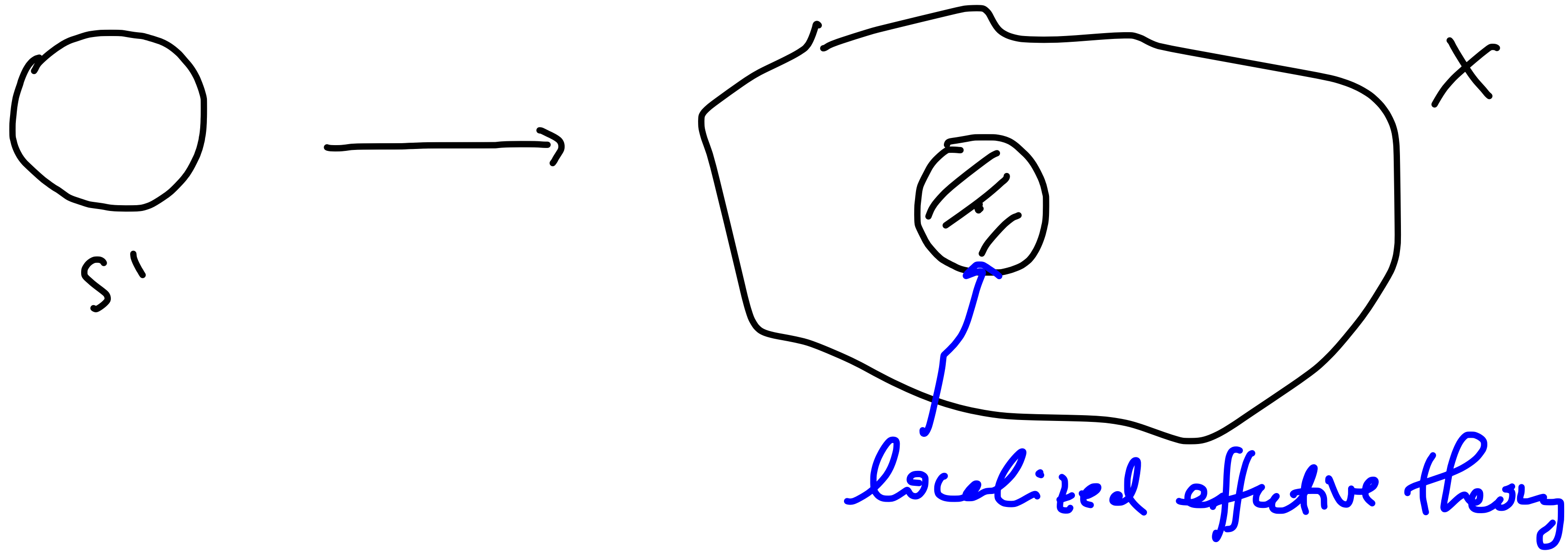
In topological QM, we find

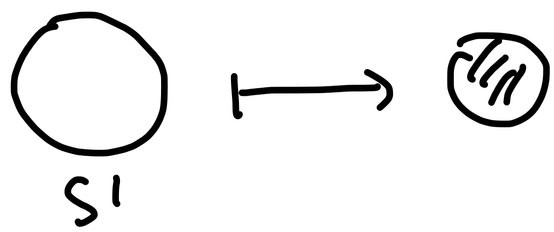
$$\int_{\mathrm{Map}(S^1, X)} e^{-S/\hbar} = \int_X (-)$$

"localize to const maps"

- LHS = analytic index
- RHS = top. index

Physics "derivation" of index theorem



Locally,  $\simeq \mathbb{R}^{2n}$

and glued on X as a family of effective field theory

This can be done rigorously within the framework of effective BV quantization

- Effective action $\rightsquigarrow \gamma$
- QME $\rightsquigarrow$ Fedosov equation
- BV integral $\rightsquigarrow$ trace map
- Partition function $\rightsquigarrow$ Algebraic index

Next time we explain this ...